Glueing terms to models

Variations on nbe.

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* NBE is a form of model construction
* It requires extra structure because we want to be able to extract syntactical objects from the model

$$\mathrm{reify}_\sigma : \mathcal{M}_\sigma \longrightarrow NF_\sigma$$

<u>Ex</u>: SK-calculus (simply-typed)

$$\sigma, \tau, \ldots ::= \mathbb{N} \mid \sigma \to \tau$$

model
$$\begin{cases} \mathcal{J}_{\mathbb{N}} = \mathbb{N} \\ \mathcal{J}_{\sigma\to\tau} = \mathcal{J}_{\sigma} \longrightarrow \mathcal{J}_{\tau} \end{cases}$$

$$
\begin{aligned}
t, u, \ldots ::= {} & K_{\sigma,\tau} : \sigma \to \tau \to \sigma \\
\mid {} & S_{\sigma,\tau,u} : (\sigma \to \tau \to u) \to (\sigma \to \tau) \to \sigma \to u \\
\mid {} & \_ \$ \_ : (\sigma \to \tau) \to \sigma \to \tau
\end{aligned}
$$

$$
\begin{aligned}
[\![\_]\!] &: Tm_\sigma \to \mathcal{T}_\sigma \\
[\![K_{\sigma,\tau}]\!] &= \lambda x. \lambda y. x \\
[\![S_{\sigma,\tau,u}]\!] &= \lambda g. \lambda f. \lambda x. g\, x\, (f\, x) \\
[\![t \$ u]\!] &= [\![t]\!]\, [\![u]\!]
\end{aligned}
$$

But how do we go from $\mathcal{T}_\sigma$ to NF_σ ?

$\Rightarrow$ Coquand & Dybjer 98 taught us how to do it.

$$t, u, \ldots ::= K_{\sigma,\tau} \mid K_{\sigma,\tau} \,\$\, t \mid S_{\sigma,\tau,\upsilon} \mid S_{\sigma,\tau,\upsilon} \,\$\, t \mid S_{\sigma,\tau,\upsilon} \,\$\, t \,\$\, u$$

$$\mathcal{T}_{\mathbb{N}} = NF_{\mathbb{N}} \qquad \mathcal{T}_{\sigma \to \tau} = NF_{\sigma \to \tau} \times \mathcal{T}_\sigma \to \mathcal{T}_\tau$$

$$[\![K_{\sigma,2}]\!] = \langle K_{\sigma,2},\ \lambda x. \langle K_{\sigma,2} \$ \mathrm{reify}_\sigma\, x,\ \lambda y. x \rangle\rangle$$

$$\begin{aligned}
[\![S_{\sigma,2,u}]\!] = \\
&\langle S_{\sigma,2,u} &&, \lambda g. \\
&\langle S_{\sigma,2,u} \$ \mathrm{reify}\, g &&, \lambda f. \\
&\langle S_{\sigma,2,u} \$ \mathrm{reify}\, g \$ \mathrm{reify}\, f, &&\lambda x. g \$ x \$ (f x) \rangle\rangle\rangle
\end{aligned}$$

$$[\![t \$ u]\!] = [\![t]\!] \$ [\![u]\!]$$

where $\langle t, f \rangle \$ x = f\, x$

$\mathrm{reify}_{\mathbb{N}}\ t = t \qquad \mathrm{reify}_{\sigma \to \tau}\langle t, f\rangle = t$

$\mathrm{norm}_\sigma : \mathrm{Tm}_\sigma \longrightarrow \mathrm{NF}_\sigma$

$\mathrm{norm}_\sigma\ t = \mathrm{reify}_\sigma\ [\![t]\!]$

Quick remark:

$$\Gamma_\sigma = NF_\sigma \times \Gamma^\star_\sigma$$

$$\Gamma^\star_{\mathbb{N}} = \mathbb{1}$$

$$\Gamma^\star_{\sigma\to\tau} = \Gamma_\sigma \to \Gamma_\tau$$

STLC $\quad t, u, \ldots ::= x \mid \lambda x.t \mid t \$ u$

NF: $t, u ::= \lambda x.t \mid m_{\mathbb{N}}$

NE: $m, n ::= x \mid m \$ t$

$$\llbracket \mathbb{N} \rrbracket = NF_{\mathbb{N}} \qquad \llbracket \sigma \to \tau \rrbracket = \llbracket \sigma \rrbracket \to \llbracket \tau \rrbracket$$

$$\mathrm{reify}_{\mathbb{N}}\, t = t \qquad \mathrm{reify}_{\sigma\to\tau}\, f = \lambda x.\, \mathrm{reify}_{\tau}(f\,(\mathrm{reflect}_{\sigma}\, x))$$

$$\mathrm{reflect}_{\mathbb{N}}\, m = m \qquad \mathrm{reflect}_{\sigma\to\tau}\, m = \lambda x.\, \mathrm{reflect}_{\tau}\,(m \$ \mathrm{reify}_{\sigma}\, x)$$

But what if we don't want the η-rule?

$$\mathcal{T}_\sigma = NE_\sigma \uplus NF_\sigma \times \mathcal{T}^\star_\sigma$$

$$\mathcal{T}^\star_{\mathbb{N}} = \underline{1}$$

$$\mathcal{T}^\star_{\sigma\to\tau} = \mathcal{T}_\sigma \to \mathcal{T}_\tau$$

$$\mathrm{reify}_\sigma(\mathrm{inj}_1\, m) = m$$

$$\mathrm{reify}_\sigma(\mathrm{inj}_2\langle t, f\rangle) = t$$

$$\mathrm{reflect}_\sigma\, m = \mathrm{inj}_1\, m$$

$$[\![\_]\!]_\rho : Tm_\sigma \longrightarrow \mathcal{V}_\sigma$$

$$[\![x]\!]_\rho = \rho(x)$$

$$[\![\lambda x.t]\!]_\rho = \langle \lambda x.\ \mathrm{reify}_2(f(\mathrm{reflect}_\sigma\, x)),\ f\rangle$$

where $f = \lambda x.\ [\![t]\!]_{(\rho, x)}$

$$[\![t \,\$\, u]\!]_\rho = [\![t]\!]_\rho \,\$\, [\![u]\!]_\rho$$

where
$$\mathrm{inj}_1\, m \,\$\, u = \mathrm{inj}_1(m \,\$\, \mathrm{reify}\, u)$$
$$\mathrm{inj}_2 \langle t, f\rangle \,\$\, u = f\, u$$

And then the usual thing:

$$\mathrm{norm}_\sigma\ t = \mathrm{reify}_\sigma\ [\![t]\!]_{[\,]}$$

decides β equality but not η.

Weak-head normal form

$\mathrm{WHNF} : t, u, \ldots ::= \lambda x. t \mid m$

$\mathrm{WHNE} : m, n, \ldots ::= x \mid m \mathbin{\$} t$

First options: use embeddings to forget some structure

$$\begin{cases} \mathrm{emb}^{NF}_{\sigma} : \mathrm{WHNF}_{\sigma} \to \mathrm{Tm}_{\sigma} \\ \mathrm{emb}^{NE}_{\sigma} : \mathrm{WHNE}_{\sigma} \to \mathrm{Tm}_{\sigma} \end{cases}$$

But what if we want $x \mathbin{\$} ((\lambda x. x) \mathbin{\$} y)$ not to be reduced to $x \mathbin{\$} y$?

⇒ We need to, somehow, be able to cancel reductions in the model!

$$\mathcal{V}_\sigma = \mathrm{Tm}_\sigma \times \underbrace{\left[\mathrm{WHNE}_\sigma \uplus \mathrm{WHNF}_\sigma \times \mathcal{V}^{\star}_\sigma \right]}_{\text{similar to previous example for an } \eta\text{-free reduction relation!}}$$

source term to cancel all reductions

$$\llbracket x \rrbracket_\rho = \rho(x)$$

$$\llbracket \lambda x.t \rrbracket_\rho = \langle \lambda x.t,\ \mathrm{inj}_2\langle \lambda x.t,\ \lambda x.\llbracket t \rrbracket(\rho, x)\rangle\rangle$$

$$\llbracket t \mathbin{\$} u \rrbracket_\rho = \llbracket t \rrbracket_\rho \mathbin{\$} \llbracket u \rrbracket_\rho$$

where

$$\langle t, \mathrm{inj}_1\, m\rangle \mathbin{\$} \langle u, -\rangle = \langle t \mathbin{\$} u,\ \mathrm{inj}_1(m \mathbin{\$} u)\rangle$$

$$\langle t, \mathrm{inj}_2(t, f)\rangle \mathbin{\$} u = \langle t \mathbin{\$} \pi_1 u,\ \pi_2(f\, u)\rangle$$

Conclusions:

- ★ Adding structure to models to facilitate reification
- ★ Deciding more exotic reduction relations than the usual $\beta\eta$

Sorry for the poor writing.

Do you have any questions?

References:

* Intuitionistic Model Constructions & Normalization proofs. Coquand, Dybjer 98

Why the hell is the radio playing "suicide is painless" whilst I'm finishing this?